

Partial Differential Equations

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FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables. The method is best illustrated by the following examples :

Example 1 *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \quad (i)$$

Solution. Differentiating (i) partially with respect to x and y , we get

$$2 \frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x} \frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$2 \frac{\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y} \frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (i), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

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Example 2 Form the partial differential equations (by eliminating the arbitrary functions) from

(a) $z = (x + y)\phi(x^2 - y^2)$ (P.T.U., 2009)

(b) $z = f(x + at) + g(x - at)$ (V.T.U., 2009) (c) $f(x^2 + y^2, z - xy) = 0$

Solution. (a) We have $z = (x + y)\phi(x^2 - y^2)$

Differentiating z partially with respect to x and y ,

$$p = \frac{\partial z}{\partial x} = (x + y)\phi'(x^2 - y^2) \cdot 2x + \phi(x^2 - y^2), \quad (i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\phi'(x^2 - y^2) \cdot (-2y) + \phi(x^2 - y^2) \quad (ii)$$

From (i), $p - \frac{z}{x+y} = 2x(x+y)\phi'(x^2 - y^2)$

From (ii), $q - \frac{z}{x+y} = -2y(x+y)\phi'(x^2 - y^2)$

Division gives $\frac{p - z/(x+y)}{q - z/(x+y)} = -\frac{x}{y}$

i.e.,

$$[p(x + y) - z]y = -[q(x + y) - z]x$$

i.e.,

$$(x + y)(py + qx) - z(x + y) = 0$$

Hence

$$py + qz = z \text{ is required equation.}$$

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(b) We have $z = f(x + at) + g(x - at)$... (i)

Differentiating z partially with respect to x and t ,

$$\frac{\partial z}{\partial x} = f'(x + at) + g'(x - at), \quad \frac{\partial^2 z}{\partial x^2} = f''(x + at) + g''(x - at) \quad \text{(ii)}$$

$$\frac{\partial z}{\partial t} = af'(x + at) - ag'(x - at), \quad \frac{\partial^2 z}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at) = a^2 \frac{\partial^2 z}{\partial x^2}$$

Thus the desired partial differential equation is $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ which is an equation of the second order and (i) is its solution.

(c) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f(u, v) = 0$.
Differentiating partially w.r.t. x and y , we have

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0$$

or

$$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + p) = 0 \quad (i)$$

and

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x + q) = 0 \quad (ii)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (i) and (ii), we get

$$\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0 \quad \text{or} \quad xq - yp = x^2 - y^2.$$

Example 3 Find the differential equation of all planes which are at a constant distance a from the origin.

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \quad (i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then
$$l^2 + m^2 + n^2 = 1 \quad \text{or} \quad n = \sqrt{1 - l^2 - m^2}$$

$\therefore$ (i) becomes
$$lx + my + \sqrt{1 - l^2 - m^2} \cdot z = a \quad \dots(ii)$$

Differentiating partially w.r.t. x , we get

$$l + \sqrt{1 - l^2 - m^2} \cdot p = 0 \quad (iii)$$

Differentiating partially w.r.t. y , we get

$$m + \sqrt{1 - l^2 - m^2} \cdot q = 0 \quad (iv)$$

Now we have to eliminate l, m from (ii), (iii) and (iv).

From (iii), $l = -\sqrt{1 - l^2 - m^2} \cdot p$ and $m = -\sqrt{1 - l^2 - m^2} \cdot q$
 Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $(l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2$ or $1 - l^2 - m^2 =$
 $1 - \frac{p^2 + q^2}{1 + p^2 + q^2} = \frac{1}{1 + p^2 + q^2}$

Also $l = -\frac{p}{\sqrt{1 + p^2 + q^2}}$ and $m = -\frac{q}{\sqrt{1 + p^2 + q^2}}$

Substituting the values of l, m and $1 - l^2 - m^2$ in (ii), we obtain

$$-\frac{px}{\sqrt{1 + p^2 + q^2}} - \frac{qy}{\sqrt{1 + p^2 + q^2}} + \frac{1}{\sqrt{1 + p^2 + q^2}} z = a$$

or $z = px + qy + a\sqrt{1 + p^2 + q^2}$ which is the required partial differential equation.

SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution

$$f(x, y, z, a, b) = 0 \quad (1)$$

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a *particular integral*.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ , which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b , if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

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EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 4 Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$.

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2 y^2 - \frac{1}{2} \cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3 y^2 - \frac{1}{4} \sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3 y^3 - \frac{1}{4} \cos(2x - y) = x \int f(y) dy + \int g(y) dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \quad \text{and} \quad \int g(y)dy = v(y).$$

Thus $z = \frac{1}{4} \cos(2x - y) - x^3 y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 5 Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Solution. If z were function of x alone, the solution would have been $z = A \sin x + B \cos x$, where A and B are constants. Since z is a function of x and y , A and B can be arbitrary functions of y .

Hence the solution of the given equation is

$$z = f(y) \sin x + \phi(y) \cos x$$

$$\therefore \frac{\partial z}{\partial x} = f(y) \cos x - \phi(y) \sin x$$

When $x = 0$; $z = e^y$, $\therefore e^y = \phi(y)$. When $x = 0$, $\frac{\partial z}{\partial x} = 1$,
 $\therefore 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 6 Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, for which $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ when y is an odd multiple of $\pi/2$.

Solution. Given equation is $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$
Integrating w.r.t. x , keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad (i)$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2 \sin y$,
 $\therefore -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$
 $\therefore$ (i) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y , keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \quad (\text{ii})$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$$\therefore 0 = 0 + 0 + g(x) \quad \text{or} \quad g(x) = 0 [\because \cos(2n+1)\pi/2 = 0]$$

Hence from (ii), the complete solution is $z = (1 + \cos x) \cos y$.

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